Aerodynamics Formulas

Definitions

p = The air pressure. ($Pa = N/m^2$)
ρ = The air density. (kg/m^3)
g = The gravitational constant. (Value at sea level is $9.81 N/kg$) (N/kg)
h = The height above the earth surface. (m)
V = The speed of the airplane relative to the air. (m/s)
p_t = The total pressure. ($Pa = N/m^2$)
p_0 = The static pressure. ($Pa = N/m^2$)
S = The wing surface. (m^2)
L = The lift force. (N)
C_L = The lift coefficient. (no unit)
D = The drag force. (N)
C_D = The drag coefficient. (no unit)

C_{D_i} = The induced drag coefficient. (no unit)
e = The Oswald factor. (Usually has a value between 0.8 and 0.9) (no unit)
A = The aspect ratio. (no unit)
b = The wing span (from left wing tip to right wing tip, so it's not just the length of one wing). (m)
D_i = The induced drag. (no unit)
C_{D_0} = The friction and pressure drag coefficient. (no unit)

M = The Mach number. (no unit)
a = The speed of sound. ($340 m/s$ at sea level) (m/s)
Re = The Reynolds number. (no unit)
L = A characteristic length. Often the length of an object. (m)
μ = The viscosity of the air. (Normal air has viscosity $17.9 \times 10^{-6} kg/(ms)$) ($kg/(ms)$)

W = The weight of the aircraft. (N)
T = The thrust of the aircraft. (N)
L_w = The wing loading. ($Pa = N/m^2$)
n = The load factor. (no unit)

Two-dimensional aerodynamics formulas

The pressure in a certain part of the atmosphere is equal to the weight of the air column on top. The formula describing this statement is known as the hydrostatic equation:

$$dp = -\rho g (dh) \tag{1}$$

An equation which looks a bit like the previous equation, is the Euler equation:

$$dp = -\rho V(dV) \tag{2}$$

So, if we integrate this equation, we find bernoulli's equation:

$$p + \frac{1}{2}\rho V^2 = C \tag{3}$$

Where C is a constant. So $p + \frac{1}{2}\rho V^2$ is constant for any 2 points along a streamline. Using this formula, the airspeed can be calculated:

$$V_0 = \sqrt{2\frac{p_t - p_0}{\rho}} \tag{4}$$

Bernoulli's equation states that $-dp = d(\frac{1}{2}\rho V^2)$. By integrating that over the wing surface, and implementing a constant, the following formula can be found:

$$L = C_L \frac{1}{2}\rho V^2 S \tag{5}$$

Similar to this, also the drag force can be calculated:

$$D = C_D \frac{1}{2}\rho V^2 S \tag{6}$$

Induced Drag

However, the previously discussed formulas work well for two-dimensional cases. In three dimensions there is also another type of drag, called the induced drag. This type of drag also has a coefficient:

$$C_{D_i} = \frac{C_L^2}{\pi A e} \tag{7}$$

But in this case, A is not known yet. A, the aspect ratio, is the relationship between the length and the width of the wing. However, the width of the wing is not constant. So by multiplying the ratio $A = \frac{wingspan}{wingwidth}$ on both sides of the fraction by the wing span, the following formula appears:

$$A = \frac{b^2}{S} \tag{8}$$

Now, using all this data (and the fact that $C_L = \frac{2L}{\rho V^2 S}$, the induced drag can be calculated:

$$D_i = C_{D_i} \frac{1}{2}\rho V^2 S = \frac{2L^2}{\rho S \pi A e}\frac{1}{V^2} \tag{9}$$

So by using the formula:

$$C_D = C_{D_0} + C_{D_i} = C_{D_0} + \frac{C_l^2}{\pi A e} \tag{10}$$

The total drag can be calculated, using equation (6).

Characteristic numbers

There are also a few numbers which characteristic the type of flow. An example is the Mach number, which is calculated using:

$$M = \frac{V}{a} \tag{11}$$

There are different names for different ranges of Mach numbers:

- $M < 0.8$: Subsonic
- $0.8 < M < 1.2$: Transonic
- $1.2 < M < 4$: Supersonic
- $4 < M$: Hypersonic

Next to the Mach number, there is also the Reynolds number:

$$Re = \frac{\rho V L}{\mu} \tag{12}$$

The Reynolds number is an indication if, and where, separation occurs. High Reynolds numbers usually result in a more turbulent flow, while low Reynolds numbers result in a more laminar flow.

Flight types

In a horizontal (no change of height) steady (no roll) straight (no yaw) flight, the following conditions must apply:

$$W = L = C_L \frac{1}{2} \rho V^2 S \tag{13}$$

$$T = D = C_D \frac{1}{2} \rho V^2 S \tag{14}$$

Divide these equations, and you will find that:

$$\frac{W}{T} = \frac{L}{D} = \frac{C_L}{C_D} \tag{15}$$

Also, it is interesting to notice that the minimal speed an airplane can have, can be calculated, if the maximum lift coefficient is known:

$$W = L = C_{L_{max}} \frac{1}{2} \rho V_{min}^2 S \tag{16}$$

There is also a factor called the wing loading. This is equal to:

$$L_w = \frac{W}{S} = C_L \frac{1}{2} \rho V^2 \tag{17}$$

However, when there is no horizontal flight, but if the airplane is climbing, some of the previous formulas don't apply. In this case, a load factor can be introduced. This can be calculated as follows:

$$n = \frac{L}{W} \tag{18}$$

So in a horizontal flight, the load factor is 1.

Since $L_{max} = C_{L_{max}} \frac{1}{2} \rho V^2 S$ and $W = C_{L_{max}} \frac{1}{2} \rho V_{min}^2 S$ it can also be derived that:

$$n_{max} = \frac{L_{max}}{W} = \left(\frac{V}{V_{min}} \right)^2 \tag{19}$$

Aircraft Limits

1 Velocities at Different Altitudes

The **flight envelope** is more or less defined as the combinations of velocity and height at which the airplane can fly in a normal way. For a certain height, an aircraft has a minimum and a maximum velocity. However, this minimum and maximum velocity differs for different heights. First let's look at the minimum flight velocity. This minimum velocity is:

$$V_{min} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}} \tag{1}$$

So if $h \uparrow$ then $\rho \downarrow$ and thus $V_{min} \uparrow$. For higher altitudes the minimum velocity increases due to a decrease in air density.

The maximum velocity depends on the power that is available. It is the velocity at which $P_{a_{max}} = P_r$. However, at different altitudes the aircraft usually has a different $P_{a_{max}}$, so it's kind of hard to calculate the maximum velocity. But this maximum velocity can be exceeded in a dive. When doing this, the airplane exits its flight envelope, which is usually considered to be a rather dangerous thing.

2 Equivalent Airspeed

The airspeed indicator of an aircraft doesn't indicate the **true airspeed** (the velocity of the aircraft with respect to the surrounding air). Instead, it indicates the **equivalent airspeed**, which is the airspeed that gives the same dynamic pressure $q = \frac{1}{2}\rho V^2$ at sea level, as the true airspeed in the current atmosphere. So at sea level the true airspeed and the equivalent airspeed are equal. But if the altitude increases, and thus the density decreases, a higher velocity is needed to reach the same dynamic pressure. Therefore the equivalent airspeed is generally lower than the true airspeed (and the difference increases with increasing altitudes). The relation between the true airspeed V and the equivalent airspeed V_e can be found as follows:

$$\frac{1}{2}\rho_0 V_e^2 = \frac{1}{2}\rho V^2 \qquad \Rightarrow \qquad V_e = V\sqrt{\frac{\rho}{\rho_0}} \tag{2}$$

where $\rho_0 = 1.225\ kg/m^3$ is the air density at sea-level. An interesting to note is that the minimum equivalent airspeed is:

$$V_{e_{min}} = V_{min}\sqrt{\frac{\rho}{\rho_0}} = \sqrt{\frac{W}{S}\frac{2}{\rho_0}\frac{1}{C_L}} \tag{3}$$

So the minimum equivalent airspeed is constant at different altitudes. This saves the pilot a lot of calculations, since the airspeed indicator of an airplane also indicates the equivalent airspeed.

3 Maximum Height

The aircraft can not fly at infinite heights. The higher you go, the less air you find, and air is something airplanes need for thrust and lift. So there must be a ceiling. This ceiling is the place at which the airplane can not go any higher. Let's define the rate of climb RC as $-RD$. So the maximum rate of climb is 0 at the ceiling. The rate of climb can be calculated using:

$$RC = \frac{P_a - P_r}{W} \tag{4}$$

So at the theoretical ceiling, when $RC_{max} = 0$ also $(P_a - P_r)_{max} = 0$. Thus $P_a \leq P_r$ and the airplane can only fly in the ceiling if $P_a = P_r$.

However, this ceiling is only a **theoretical ceiling**. Since if the rate of climb is 0 m/s in the theoretical ceiling, how could you get there? There is also a **service ceiling**, which is in practice about the highest point at which aircrafts can fly. The service ceiling is the height at which the maximum rate of climb of the airplane is 0.5 m/s.

4 Supersonic Limits

When an aircraft is flying at supersonic velocities, shock waves occur. The shape of the shock wave can be either oblique or blunt. Oblique shock waves are caused by sharp edges and are relatively weak, while blunt shock waves are caused by rounded edges and are relatively strong. Oblique shock waves have an angle, called the **Mach angle**, which can be calculated using:

$$\mu = \arcsin \frac{a}{V} = \arcsin \frac{1}{M} \tag{5}$$

When the air passes through a shock wave, a lot of things happen. To make a list: $V \downarrow$, $p \uparrow$, $T \uparrow$, $M \downarrow$, $s \uparrow$. The entropy s increases due to a loss in energy, which is caused by additional drag called **wave drag**. This wave drag is also caused by shock waves.

When flying at high Mach numbers, buffeting can occur. This can be dangerous, and to prevent this, the airworthiness regulations define a **maximum Mach number** M_D for an airplane after several tests. This results in a maximum velocity of:

$$V_D = M_D \sqrt{\gamma R T} \tag{6}$$

Since the temperature deceases as the height increases, also the maximum velocity due to the maximum Mach number decreases as the height increases (until the stratosphere is reached where T is constant). To increase safety even more, an extra margin gets taken into account, which results in the **maximum operating Mach number** M_{M0}. This is the highest Mach number at which the aircraft is allowed to fly.

5 Gusts

If an aircraft encounters a sudden upward gust, the angle of attack (with respect to the airflow) will increase. If the air gusts travels upward with a velocity u, the change in angle of attack is:

$$\Delta\alpha = \tan \frac{u}{V} \approx \frac{u}{V} \tag{7}$$

The change in lift coefficient now is:

$$\Delta C_L = \frac{dC_L}{d\alpha} \Delta\alpha = \frac{dC_L}{d\alpha} \frac{u}{V} \tag{8}$$

This makes the change in lift the following:

$$\Delta L = (\Delta C_L) \frac{1}{2} \rho \left(V^2 + u^2\right) S = \frac{dC_L}{d\alpha} \frac{1}{2} \rho \left(uV + 2u^2 + \frac{u^3}{V}\right) S \approx \frac{dC_L}{d\alpha} \frac{1}{2} \rho u V S \tag{9}$$

In the last step the assumption was made that $u << V$. Since a sudden huge increase in lift can be dangerous (high G-forces and breaking wings may occur), the airworthiness regulations have set another limit, being the **maximum equivalent airspeed due to gust loading** V_{e_d} $[m/s]$. This makes the maximum airspeed due to gust loading:

$$V_D = V_{e_d} \sqrt{\rho} \rho_0 \tag{10}$$

However, to increase safety, there is an additional margin to this maximum allowed airspeed, being the **maximum operational velocity** V_{M0}. This is the highest velocity at which an aircraft is allowed to fly.

6 Limit Overview

Next to the limits we just saw, there is one additional limit to the flight envelope of an airplane. This is the maximum pressure difference. The pressure cabin can only take a maximum pressure difference, which may not be exceeded. This is the last limit that will be discussed.

It's time to make a graph out of all the limits we have just talked about. This graph can be seen in figure 1. It gives an impression on the flight envelope of a normal aircraft.

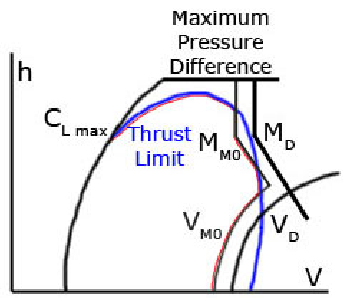

Figure 1: Visualization of the flight envelope.